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Equivalence transformations and differential invariants of a generalized nonlinear Schrödinger equation

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Abstract

By using Lie's invariance infinitesimal criterion, we obtain the continuous equivalence transformations of a class of nonlinear Schrödinger equations with variable coefficients. We construct the differential invariants of order 1 starting from a special equivalence subalgebra $\mathcal{E}_{\chi_0}$. We apply these latter ones to find the most general subclass of variable coefficient nonlinear Schrödinger equations which can be mapped, by means of an equivalence transformation of $\mathcal{E}_{\chi_0}$, to the well-known cubic Schrödinger equation. We also provide the explicit form of the transformation.

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1. Introduction

In this paper we investigate the equivalence transformations and the differential invariants associated with the following $(1+1)$-dimensional generalized nonlinear Schrödinger equation

$$\mathrm{i}u_t + f(t,x)u_{xx} + g(t,x)|u|^2u + h(t,x)u = 0, \tag{1.1}$$

where $f(t,x)$, $g(t,x)$ and $h(t,x)$ are real functions of t and x, and the subscript denotes partial differentiation with respect to that variable.

The complex function $u(t,x)$ has different physical meanings in different branches of physics. Winternitz and Gagnon in [1] have shown a wide range of detailed applications of relevant physical interest in which equation (1.2) is involved. Moreover, a wide subclass of equations considered in [2] belongs to the class (1.1).

In the case $f, g = 1$ and $h = 0$, equation (1.1) reduces to the well-known nonlinear Schrödinger (NLS) equation

$$\mathrm{i}u_t + u_{xx} + |u|^2u = 0. \tag{1.2}$$

Equation (1.2) arises in several branches of physics including nonlinear optics, condensed matter physics, hydrodynamics and so on (for the derivation and wide applications of equation (1.2), refer [3–7]). It has been shown that the NLS equation is one of the completely integrable nonlinear partial differential equations (PDEs) in (1+1) dimensions and admits several interesting mathematical properties including infinite number of conservation laws, Lie–Bäcklund symmetries, N-soliton solutions and so on.

An equivalence transformation for the family (1.1) is a non-degenerate transformation of dependent and independent variables mapping (1.1) to another equation of the same family but with different functions, say $(\hat{f}, \hat{g}, \hat{h})$, from the original ones. Thus, solutions of an equation can be transformed to the solutions of an equivalent equation. The main advantage of this procedure is that instead of solving individual equations, one can develop schemes for complete equivalent classes. One may note that the classical Lie symmetry transformations are nothing but special subgroups of these equivalence groups of transformations since Lie symmetry transformations map an equation into itself. Winternitz and Gagnon in [1] investigated the fibre-preserving equivalence transformations of equation (1.1) by using the finite form. Their research was essentially devoted to obtain symmetries and some exact solutions.

Recently, several works have been devoted to study the equivalence transformations of certain important nonlinear dynamical systems (see for a review [8, 9] and bibliography therein).

One of the classical studies in the theory of differential equations is finding differential invariants. The origin dates back to Laplace who derived two invariants for linear hyperbolic equations. These invariants are now called Laplace invariants. The generalization of these invariants to elliptic and hyperbolic equations was derived by Cotton (for historical notes and recent developments, one may refer [10, 11]). Recently, a renewal of interest has been initiated to study the differential invariants of the equivalence algebra of certain multidimensional linear PDEs as well as nonlinear PDEs [12–26].

In this paper, by using the infinitesimal Lie criterion [27], we explore both the equivalence transformation algebra $\mathcal{E}$ of equation (1.1) and some of the differential invariants of a special equivalence subalgebra $\mathcal{E}_{\chi_0}$. More specifically by treating the functions f, g and h as arbitrary parameters, we explore the most general nonlinear PDE of the form (1.1) which can be mapped by an equivalence transformation of the subalgebra $\mathcal{E}_{\chi_0}$ to the standard NLS equation. Through this analysis, we bring out a family of integrable variable coefficient NLS equations that can be mapped to the standard NLS equation. We also construct the transformation that connects the variable coefficient NLS equation to the standard NLS equation.

The plan of the paper is as follows. In section 2, we get the infinitesimal generators of the equivalence transformation algebra $\mathcal{E}$ of equation (1.1). In section 3, we investigate the differential invariants of (1.1) associated with the subalgebra $\mathcal{E}_{\chi_0}$ and show that the latter one admits first-order differential invariants. As an application of these differential invariants, in section 4, we derive the functional forms of f, g and h which characterize the subclass of equation (1.1) that can be transformed to a standard NLS equation (1.2). We present our conclusions in section 5.

2. Equivalence transformations

In order to study the equivalence transformations of equation (1.1), we rewrite complex equation (1.1) as a system of real equations by introducing a transformation $u = v + \mathrm{i}w$, that is,

$$v_t + f(t,x)w_{xx} + g(t,x)(v^2 + w^2)w + h(t,x)w = 0, \qquad (2.1a)$$

$$w_t - f(t,x)v_{xx} - g(t,x)(v^2+w^2)v - h(t,x)v = 0. \tag{2.1b}$$

It is worth noting that in the family of the above system fall some classes of reaction–diffusion models (see e.g. [28, 29] and references therein).

An equivalence transformation of the system (2.1*a*)–(2.1*b*) is a non-degenerate change of variables from (t,x,v,w) to $(\hat{t},\hat{x},\hat{v},\hat{w})$ and transforming the equations of the form (2.1*a*)–(2.1*b*) into another system of the same form, but with different functions $\hat{f}(\hat{t},\hat{x})$, $\hat{g}(\hat{t},\hat{x})$ and $\hat{h}(\hat{t},\hat{x})$. The equivalence transformations for our system (2.1*a*)–(2.1*b*) are obtained by making use of Lie's infinitesimal criterion [27]. However, in the case of the infinitesimal equivalence generator, we demand not only the invariance of (2.1*a*)–(2.1*b*) but also the invariance of the so-called *auxiliary conditions* in the *augmented space* (t,x,v,w,f,g,h). In other words, one also needs to consider the following conditions, in addition to the usual invariance conditions:

$$f_v = f_w = 0, \qquad g_v = g_w = 0, \qquad h_v = h_w = 0, \tag{2.2}$$

which characterize the functional dependence of the functions f, g and h.

Let us consider the one-parameter group of equivalence transformations, G_ε, in the *augmented space* (t,x,v,w,f,g,h) given by

$$\hat{t} = t + \varepsilon\xi^1(t,x,v,w) + \mathcal{O}(\varepsilon^2), \tag{2.3a}$$
$$\hat{x} = x + \varepsilon\xi^2(t,x,v,w) + \mathcal{O}(\varepsilon^2), \tag{2.3b}$$
$$\hat{v} = v + \varepsilon\eta^1(t,x,v,w) + \mathcal{O}(\varepsilon^2), \tag{2.3c}$$
$$\hat{w} = w + \varepsilon\eta^2(t,x,v,w) + \mathcal{O}(\varepsilon^2), \tag{2.3d}$$
$$\hat{f} = f + \varepsilon\nu^1(t,x,v,w,f,g,h) + \mathcal{O}(\varepsilon^2), \tag{2.3e}$$
$$\hat{g} = g + \varepsilon\nu^2(t,x,v,w,f,g,h) + \mathcal{O}(\varepsilon^2), \tag{2.3f}$$
$$\hat{h} = h + \varepsilon\nu^3(t,x,v,w,f,g,h) + \mathcal{O}(\varepsilon^2), \tag{2.3g}$$

where ε is the infinitesimal group parameter. The vector field associated with the infinitesimal equivalence transformations (2.3*a*)–(2.3*g*) can be written as

$$Y = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta^1\frac{\partial}{\partial v} + \eta^2\frac{\partial}{\partial w} + \nu^1\frac{\partial}{\partial f} + \nu^2\frac{\partial}{\partial g} + \nu^3\frac{\partial}{\partial h}. \tag{2.4}$$

Since (2.1*a*)–(2.1*b*) involve second derivatives, we need to consider second prolongation of the operator Y. Before proceeding further, we introduce the notation

$$\begin{aligned}&(f^1,f^2,f^3) \equiv (f,g,h), \qquad (x^1,x^2) \equiv (t,x), \qquad (y^1,y^2) \equiv (v,w),\\ &y^i_j = \frac{\partial y^i}{\partial x^j}, \qquad y^i_{jk} = \frac{\partial^2 y^i}{\partial x^j \partial x^k}, \qquad j,i,k = 1,2.\end{aligned} \tag{2.5}$$

With the above notations, the second prolongation of the operator Y can be written as

$$Y^{(2)} = Y + \zeta^i_j\frac{\partial}{\partial y^i_j} + \zeta^i_{jj}\frac{\partial}{\partial y^i_{jj}} + \tilde{\omega}^r_j\frac{\partial}{\partial f^r_{x^j}} + \bar{\omega}^r_j\frac{\partial}{\partial f^r_{y^j}}, \qquad r = 1,2,3, \tag{2.6}$$

where the coefficients ζ^i_j and ζ^i_{jj} are given by

$$\zeta^i_j = D_j\eta^i - y^i_k D_j\xi^k, \qquad \zeta^i_{jj} = D_j\zeta^i_j - y^i_{jk}D_j\xi^k,$$

with

$$D_j = \frac{\partial}{\partial x^j} + y^i_j\frac{\partial}{\partial y^i} + y^i_{jk}\frac{\partial}{\partial y^i_k}. \tag{2.7}$$

The remaining coefficients in (2.6) are obtained through the following prolongation formula:

$$\tilde{\omega}^r_j = \tilde{D}_j(\nu^r) - f^r_{x^k}\tilde{D}_j(\xi^k) - f^r_{y^i}\tilde{D}_j(\eta^i), \tag{2.8}$$

where

$$\tilde{D}_j = \frac{\partial}{\partial x^j} + f^r_{x^j}\frac{\partial}{\partial f^r}, \qquad \bar{D}_j = \frac{\partial}{\partial y^j} + f^r_{y^j}\frac{\partial}{\partial f^r}. \tag{2.9}$$

The invariance of equations (2.1*a*)–(2.1*b*) under the one-parameter group of equivalence transformations (2.3*a*)–(2.3*g*) can be written as [27]

$$Y^{(2)}(v_t + f(t,x)w_{xx} + g(t,x)(v^2+w^2)w + h(t,x)w) = 0, \tag{2.10}$$

$$Y^{(2)}(w_t - f(t,x)v_{xx} - g(t,x)(v^2+w^2)v - h(t,x)v) = 0, \tag{2.11}$$

$$Y^{(2)}(f_v) = Y^{(2)}(f_w) = Y^{(2)}(g_v) = Y^{(2)}(g_w) = Y^{(2)}(h_v) = Y^{(2)}(h_w) = 0, \tag{2.12}$$

under the constraints that the variables v, w, f, g and h have to satisfy equations (2.1*a*)–(2.1*b*) and (2.2).

Substituting the second prolongation (2.6) into (2.10)–(2.12) and solving the resultant equations (*determining system*), we find

$$\xi^1 = \varphi(t), \qquad \xi^2 = \psi(x), \tag{2.13}$$

$$\eta^1 = \left(a + \frac{\psi_x}{2}\right)v + \chi(t)w, \qquad \eta^2 = \left(a + \frac{\psi_x}{2}\right)w - \chi(t)v, \tag{2.14}$$

$$\nu^1 = (2\psi_x - \varphi_t)f, \qquad \nu^2 = -(\psi_x + \varphi_t + 2a)g, \qquad \nu^3 = -\left(h\varphi_t + \tfrac{1}{2}f\psi_{xxx} + \chi_t\right), \tag{2.15}$$

$\chi(t)$, $\varphi(t)$ and $\psi(x)$ are arbitrary functions of their arguments and a is an arbitrary constant.

The associated equivalence algebra $\mathcal{E}$ is an infinite dimensional one and is generated by the operators

$$Y_a = v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w} - 2g\frac{\partial}{\partial g}, \tag{2.16}$$

$$Y_\chi = \chi w\frac{\partial}{\partial v} - \chi v\frac{\partial}{\partial w} - \chi_t\frac{\partial}{\partial h}, \tag{2.17}$$

$$Y_\varphi = \varphi\frac{\partial}{\partial t} - f\varphi_t\frac{\partial}{\partial f} - g\varphi_t\frac{\partial}{\partial g} - h\varphi_t\frac{\partial}{\partial h}, \tag{2.18}$$

$$Y_\psi = \psi\frac{\partial}{\partial x} + \frac{1}{2}v\psi_x\frac{\partial}{\partial v} + \frac{1}{2}w\psi_x\frac{\partial}{\partial w} + 2f\psi_x\frac{\partial}{\partial f} - g\psi_x\frac{\partial}{\partial g} - \frac{1}{2}f\psi_{xxx}\frac{\partial}{\partial h}. \tag{2.19}$$

In the following, for the sake of simplicity and in view of our application, we restrict to a subalgebra $\mathcal{E}_{\chi_0}$ of $\mathcal{E}$, by assuming $\chi(t) = \chi_0 =$ constant.

3. Differential invariants of the equivalence subalgebra $\mathcal{E}_{\chi_0}$

Differential invariants of order n of the equivalence subalgebra $\mathcal{E}_{\chi_0}$ are not only functions of the independent variables t and x but also functions of f, g, h and their derivatives up to the maximal order n and invariant with respect to the equivalence operator of the subalgebra $\mathcal{E}_{\chi_0}$. In this subalgebra, the operator (2.17) reduces to

$$Y_{\chi_0} = w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w}, \tag{3.1}$$

and the differential invariants of every order result invariant with respect to it. For this reason, we will take into consideration only the operators (2.16), (2.18) and (2.19).

3.1. Differential invariants of order zero

First, let us seek differential invariants of order zero of the form

$$J = J(t, x, f, g, h). \tag{3.2}$$

Applying the invariant test $Y(J) = 0$ to the operators Y_a, Y_φ with $\varphi = 1$ and Y_ψ with $\psi = 1$, we find $J = J(f, h)$. Since ψ_x and ψ_{xxx} are functionally independent, the invariant test $Y_\psi(J) = 0$ provides us the following two equations:

$$\frac{\partial J}{\partial f} = 0, \qquad \frac{\partial J}{\partial h} = 0. \tag{3.3}$$

As a result equations (2.1*a*)–(2.1*b*) do not admit any differential invariant of order zero. In the following we seek higher order differential invariants, if any.

3.2. Differential invariants of first order

To obtain differential invariants of first order

$$J = J(t, x, f, g, h, f_t, f_x, g_t, g_x, h_t, h_x), \tag{3.4}$$

in which the invariant includes both the spatial and time derivatives of the functions f, g and h, we consider the first prolongation of the operator Y given by

$$Y^{(1)} = Y + \tilde{\omega}_j^r \frac{\partial}{\partial f^r_{x^j}}, \tag{3.5}$$

where the coefficients $\tilde{\omega}_j^r$ can be constructed from (2.8). The explicit forms of the first prolongations of the generators $Y_a^{(1)}$, $Y_\varphi^{(1)}$ and $Y_\psi^{(1)}$ are given by

$$Y_a^{(1)} = v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w} - 2g\frac{\partial}{\partial g} - 2g_x\frac{\partial}{\partial g_x} - 2g_t\frac{\partial}{\partial g_t}, \tag{3.6}$$

$$\begin{aligned} Y_\varphi^{(1)} = {} & \varphi\frac{\partial}{\partial t} - f\varphi_t\frac{\partial}{\partial f} - g\varphi_t\frac{\partial}{\partial g} - h\varphi_t\frac{\partial}{\partial h} - f_x\varphi_t\frac{\partial}{\partial f_x} - (2f_t\varphi_t + f\varphi_{tt})\frac{\partial}{\partial f_t}, \\ & - g_x\varphi_t\frac{\partial}{\partial g_x} - (2g_t\varphi_t + g\varphi_{tt})\frac{\partial}{\partial g_t} - h_x\varphi_t\frac{\partial}{\partial h_x} - (2h_t\varphi_t + h\varphi_{tt})\frac{\partial}{\partial h_t}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} Y_\psi^{(1)} = {} & \psi\frac{\partial}{\partial x} + \frac{v}{2}\psi_x\frac{\partial}{\partial v} + \frac{w}{2}\psi_x\frac{\partial}{\partial w} + 2f\psi_x\frac{\partial}{\partial f} - g\psi_x\frac{\partial}{\partial g} - \frac{f}{2}\psi_{xxx}\frac{\partial}{\partial h} \\ & + (2f\psi_{xx} + f_x\psi_x)\frac{\partial}{\partial f_x} + 2f_t\psi_x\frac{\partial}{\partial f_t} - (g\psi_{xx} + 2g_x\psi_x)\frac{\partial}{\partial g_x} \\ & - g_t\psi_x\frac{\partial}{\partial g_t} - \left(\frac{f}{2}\psi_{xxxx} + \frac{f_x}{2}\psi_{xxx} + h_x\psi_x\right)\frac{\partial}{\partial h_x} - \frac{f_t}{2}\psi_{xxx}\frac{\partial}{\partial h_t}. \end{aligned} \tag{3.8}$$

The differential invariant test concerned with the function J given by (3.4) reads

$$Y_\varphi^{(1)}(J) = 0, \qquad Y_\psi^{(1)}(J) = 0, \qquad Y_a^{(1)}(J) = 0. \tag{3.9}$$

Since the arbitrary functions φ and ψ and their derivatives are to be treated functionally independent, equation (3.9) can be split into the following conditions:

$$\frac{\partial J}{\partial t} = 0, \qquad \frac{\partial J}{\partial x} = 0, \qquad \frac{\partial J}{\partial h_x} = 0, \tag{3.10}$$

$$f\frac{\partial J}{\partial h} + f_t\frac{\partial J}{\partial h_t} = 0, \tag{3.11}$$

$$2f\frac{\partial J}{\partial f_x} - g\frac{\partial J}{\partial g_x} = 0, \tag{3.12}$$

$$f\frac{\partial J}{\partial f_t} + g\frac{\partial J}{\partial g_t} + h\frac{\partial J}{\partial h_t} = 0, \tag{3.13}$$

$$g\frac{\partial J}{\partial g} + g_x\frac{\partial J}{\partial g_x} + g_t\frac{\partial J}{\partial g_t} = 0, \tag{3.14}$$

$$4f\frac{\partial J}{\partial f} - 2g\frac{\partial J}{\partial g} + 2f_x\frac{\partial J}{\partial f_x} + 4f_t\frac{\partial J}{\partial f_t} - 4g_x\frac{\partial J}{\partial g_x} - 2g_t\frac{\partial J}{\partial g_t} = 0, \tag{3.15}$$

$$f\frac{\partial J}{\partial f} + g\frac{\partial J}{\partial g} + h\frac{\partial J}{\partial h} + f_x\frac{\partial J}{\partial f_x} + 2f_t\frac{\partial J}{\partial f_t} + g_x\frac{\partial J}{\partial g_x} + 2g_t\frac{\partial J}{\partial g_t} + 2h_t\frac{\partial J}{\partial h_t} = 0. \tag{3.16}$$

We note that equation (3.15) can be simplified, with the use of (3.14), to

$$2f\frac{\partial J}{\partial f} + f_x\frac{\partial J}{\partial f_x} + 2f_t\frac{\partial J}{\partial f_t} - g_x\frac{\partial J}{\partial g_x} = 0. \tag{3.17}$$

As a result, it is sufficient to solve equations (3.10)–(3.14), (3.16) and (3.17) instead of equations (3.10)–(3.16).

Equation (3.10) simplifies the differential invariant (3.4) to the form

$$J = J(f, g, h, f_t, f_x, g_t, g_x, h_t). \tag{3.18}$$

Solving the characteristic equation associated with (3.11),

$$\frac{\mathrm{d}h}{f} = \frac{\mathrm{d}h_t}{f_t}, \tag{3.19}$$

we get

$$\lambda_1 = \frac{hf_t}{f} - h_t. \tag{3.20}$$

Hence, the function J becomes $J = J(f, g, f_t, f_x, g_t, g_x, \lambda_1)$.

Similarly from (3.12) we obtain

$$\lambda_2 = \frac{g}{f}f_x + 2g_x, \tag{3.21}$$

and so the differential invariant reduces to

$$J = J(f, g, f_t, g_t, \lambda_1, \lambda_2). \tag{3.22}$$

Substituting (3.22) into (3.13) and simplifying the resultant equation, we arrive at

$$f\frac{\partial J}{\partial f_t} + g\frac{\partial J}{\partial g_t} = 0, \tag{3.23}$$

from which we get

$$\lambda_3 = \frac{g}{f}f_t - g_t. \tag{3.24}$$

As a consequence, (3.22) can be further reduced to $J = J(f, g, \lambda_1, \lambda_2, \lambda_3)$. Substituting the latter into (3.17), we obtain

$$2f\frac{\partial J}{\partial f} - \lambda_2\frac{\partial J}{\partial \lambda_2} = 0, \tag{3.25}$$

which upon integration yields the invariant

$$\beta = f\lambda_2^2. \tag{3.26}$$

Therefore, the differential invariant J assumes the form

$$J = J(g, \lambda_1, \lambda_3, \beta). \tag{3.27}$$

Now substituting (3.27) into (3.14), we get

$$g\frac{\partial J}{\partial g} + \lambda_3\frac{\partial J}{\partial \lambda_3} + 2\beta\frac{\partial J}{\partial \beta} = 0. \tag{3.28}$$

The characteristic equations associated with the PDE (3.28) can be written as

$$\frac{\mathrm{d}g}{g} = \frac{\mathrm{d}\lambda_3}{\lambda_3} = \frac{\mathrm{d}\beta}{2\beta}. \tag{3.29}$$

The invariant associated with equation (3.29) can be easily found to be of the form

$$\alpha_1 = \frac{\lambda_3}{g}, \qquad \alpha_2 = \frac{\beta}{g^2}. \tag{3.30}$$

At this point, we have

$$J = J(\lambda_1, \alpha_1, \alpha_2) \tag{3.31}$$

and equation (3.16) left unsolved. Substituting (3.31) into (3.16) and simplifying it, we arrive at

$$2\lambda_1\frac{\partial J}{\partial \lambda_1} + \alpha_1\frac{\partial J}{\partial \alpha_1} + \alpha_2\frac{\partial J}{\partial \alpha_2} = 0. \tag{3.32}$$

By integrating the characteristic equations associated with the PDE (3.32), we arrive at the following result.

Theorem 1. *The general form of the first-order differential invariants of the subalgebra $\mathcal{E}_{\chi_0}$ of equations (2.1a)–(2.1b) (or equation (1.1)), when $\lambda_1 \neq 0$, is*

$$J = J(\gamma_1, \gamma_2), \tag{3.33}$$

where γ_1 and γ_2 are two independent invariants and their explicit forms read

$$\gamma_1 = \frac{\alpha_1^2}{\lambda_1} = \frac{(gf_t - g_t f)^2}{fg^2(hf_t - h_t f)}, \qquad \gamma_2 = \frac{\alpha_2^2}{\lambda_1} = \frac{(gf_x + 2g_x f)^4}{fg^4(hf_t - h_t f)}. \tag{3.34}$$

If $\lambda_1 = 0$, the corresponding form of equations (2.1a)–(2.1b) (or equation (1.1)) should be considered separately. It is now easy to show that the cases $\alpha_1 = 0$ and $\alpha_2 = 0$ are also exceptional. Therefore, we can state that the equations $\lambda_1 = 0, \alpha_1 = 0$ and $\alpha_2 = 0$ are invariant with respect to the equivalence subalgebra $\mathcal{E}_{\chi_0}$.

4. Application of differential invariants

As stated earlier, our motivation is to map an equation of the form (1.1) to the standard NLS equation through equivalence transformations. To do this, we make use of the differential invariants (or the invariant equations), which we derived in the previous section. In particular, we consider the following equation as target one:

$$\mathrm{i}\hat{u}_{\hat{t}} + k_1\hat{u}_{\hat{x}\hat{x}} + k_2|\hat{u}|^2\hat{u} = 0, \tag{4.1}$$

where k_1 and k_2 are positive constants. One can easily check that equation (4.1) can be transformed into (1.2).

We observe that equation (4.1) does not have any first-order differential invariant, but admits invariant equations $\lambda_1 = 0, \alpha_1 = 0$ and $\alpha_2 = 0$. This observation leads us to formulate a necessary condition for an equation belonging to the class (1.1) that can be mapped through an equivalence transformation of $G_{\mathcal{E}_{\chi_0}}$ into (4.1). The condition is that the functions f, g and h must satisfy the following three equations:

$$gf_t - fg_t = 0, \qquad hf_t - fh_t = 0, \qquad gf_x + 2fg_x = 0. \tag{4.2}$$

The most general forms of f, g and h satisfying (4.2) are

$$f = f_0 \frac{n(t)}{l^2(x)}, \qquad g = g_0 n(t) l(x), \qquad h = n(t) m(x), \qquad f_0, g_0, n(t), l(x) \neq 0, \tag{4.3}$$

where $l(x)$, $m(x)$ and $n(t)$ are real functions, while f_0 and g_0 are real constants. The above forms fix equation (1.1) of the form

$$\mathrm{i}u_t + f_0 \frac{n(t)}{l^2(x)} u_{xx} + g_0 n(t) l(x) |u|^2 u + n(t) m(x) u = 0. \tag{4.4}$$

Equation (4.4) can be further transformed into

$$\mathrm{i}u_{\hat{t}} + f_0 \frac{1}{l^2(x)} u_{xx} + g_0 l(x) |u|^2 u + m(x) u = 0 \tag{4.5}$$

by means of the equivalence transformation of $G_{\mathcal{E}_{\chi_0}}$,

$$\hat{t} = \int^t \varphi(s)\,\mathrm{d}s, \qquad \hat{x} = x, \quad \hat{u} = u, \tag{4.6}$$

with $\varphi(t) = n(t)$.

To transform (4.5) further into (4.1), we need to calculate the second-order differential invariants of (4.5) or its associated real system

$$v_{\hat{t}} + f_0 \frac{1}{l^2(x)} w_{xx} + g_0 l(x)(v^2 + w^2) w + m(x) w = 0, \tag{4.7}$$

$$w_{\hat{t}} - f_0 \frac{1}{l^2(x)} v_{xx} - g_0 l(x)(v^2 + w^2) v - m(x) v = 0. \tag{4.8}$$

We look for the functions of the form

$$J = J(\hat{t}, x, l, m, l_x, m_x, l_{xx}, m_{xx}), \tag{4.9}$$

which are invariant with respect to the infinitesimal equivalence generators of (4.7) and (4.8), with the auxiliary conditions $l_{\hat{t}} = m_{\hat{t}} = 0$,

$$\Upsilon = \bar{\xi}^1 \partial_{\hat{t}} + \bar{\xi}^2 \partial_x + \bar{\eta}^1 \partial_v + \bar{\eta}^2 \partial_w + \mu^1 \partial_l + \mu^2 \partial_m \tag{4.10}$$

and its appropriate prolongations.

Introducing additional change of variables

$$f = \frac{f_0}{l^2}, \qquad g = g_0 l, \qquad h = m, \tag{4.11}$$

and following the procedure adopted in [30] for the change of variables from the old coordinates $\xi^1, \xi^2, \eta^1, \eta^2, \nu^1, \nu^2, \nu^3$ of the generator Y to the new coordinates $\bar{\xi}^1, \bar{\xi}^2, \bar{\eta}^1, \bar{\eta}^2, \mu^1, \mu^2$ of Υ, we obtain

$$\bar{\xi}^1 = -\frac{4}{3} a \hat{t} + a_0, \qquad \bar{\xi}^2 = \xi^2 = \psi(x),$$

$$\bar{\eta}^1 = \eta^1 = \left(a + \frac{\psi_x}{2}\right) v + \chi_0 w, \qquad \bar{\eta}^2 = \eta^2 = \left(a + \frac{\psi_x}{2}\right) w - \chi_0 v,$$

$$\mu^1 = -\left(\psi_x + \frac{2}{3} a\right) l, \qquad \mu^2 = -\frac{f_0}{2l^2} \psi_{xxx} + \frac{4}{3} a m,$$

with a_0 an arbitrary constant.

Considering the prolongation formula

$$\Upsilon^{(2)} = \Upsilon + \omega_x^1 \partial_{l_x} + \omega_x^2 \partial_{m_x} + \omega_{xx}^1 \partial_{l_{xx}} + \omega_{xx}^2 \partial_{m_{xx}}, \tag{4.12}$$

where

$$\begin{aligned}
\omega_x^1 &= \frac{\partial \mu^1}{\partial x} + l_x \frac{\partial \mu^1}{\partial l} + m_x \frac{\partial \mu^1}{\partial m} - l_x \frac{\partial \bar{\xi}^2}{\partial x}, \\
\omega_x^2 &= \frac{\partial \mu^2}{\partial x} + l_x \frac{\partial \mu^2}{\partial l} + m_x \frac{\partial \mu^2}{\partial m} - m_x \frac{\partial \bar{\xi}^2}{\partial x}, \\
\omega_{xx}^1 &= \frac{\partial \omega_x^1}{\partial x} + l_x \frac{\partial \omega_x^1}{\partial l} + m_x \frac{\partial \omega_x^1}{\partial m} + l_{xx} \frac{\partial \omega_x^1}{\partial l_x} + m_{xx} \frac{\partial \omega_x^1}{\partial m_x} - l_{xx} \frac{\partial \bar{\xi}^2}{\partial x}, \\
\omega_{xx}^2 &= \frac{\partial \omega_x^2}{\partial x} + l_x \frac{\partial \omega_x^2}{\partial l} + m_x \frac{\partial \omega_x^2}{\partial m} + l_{xx} \frac{\partial \omega_x^2}{\partial l_x} + m_{xx} \frac{\partial \omega_x^2}{\partial m_x} - m_{xx} \frac{\partial \bar{\xi}^2}{\partial x}
\end{aligned}$$

and repeating the same procedure followed in section 4, we find that the system (4.7)–(4.8) (or equation (4.5)) does not possess any second-order differential invariants, but admits the following invariant equation:

$$m + \frac{1}{2} f_0 \left(\frac{3}{2} \frac{l_x^2}{l^4} - \frac{l_{xx}}{l^3} \right) = 0. \tag{4.13}$$

By observing that (4.13) is also an invariant equation of (4.1), we can conclude that the necessary condition for equation (1.1) to be mapped into (4.1) through the equivalence transformation of $G_{\mathcal{E}}$ is that the functions f, g and h must be linked by the following relations:

$$f = f_0 \frac{n(t)}{l^2(x)}, \qquad g = g_0 n(t) l(x), \qquad h = \frac{1}{2} f_0 n(t) \left(\frac{l_{xx}}{l^3} - \frac{3}{2} \frac{l_x^2}{l^4} \right). \tag{4.14}$$

Now, we verify whether conditions (4.14) are also sufficient or not.

To do this, let us consider the equivalence transformation of $G_{\mathcal{E}_{\chi_0}}$, that is,

$$\hat{t} = \int^t \varphi(s)\,\mathrm{d}s, \qquad \hat{x} = \int^x \psi(s)\,\mathrm{d}s, \qquad \hat{u}(\hat{t}, \hat{x}) = u(t, x)\sqrt{\psi(x)}, \qquad \psi(x) > 0 \tag{4.15}$$

and substituting it into equation

$$\mathrm{i}u_t + f_0 \frac{n(t)}{l^2(x)} u_{xx} + g_0 n(t) l(x) |u|^2 u + \frac{1}{2} f_0 n(t) \left(\frac{l_{xx}}{l^3} - \frac{3}{2} \frac{l_x^2}{l^4} \right) u = 0, \tag{4.16}$$

we get

$$\begin{aligned}
\varphi(t)\mathrm{i}\hat{u}_{\hat{t}} &+ f_0 \frac{n(t)\psi^2(x)}{l^2(x)} \hat{u}_{\hat{x}\hat{x}} + g_0 \frac{n(t) l(x)}{\psi(x)} |\hat{u}|^2 \hat{u} \\
&+ \frac{1}{2} f_0 n(t) \left(-\frac{\psi_{xx}}{\psi^3} + \frac{3}{2} \frac{\psi_x^2}{\psi^4} + \frac{l_{xx}}{l^3} - \frac{3}{2} \frac{l_x^2}{l^4} \right) \hat{u} = 0.
\end{aligned} \tag{4.17}$$

Equation (4.17), with $f_0 = k_1$ and $g_0 = k_2$, reduces to (4.1) when the transformation (4.15) satisfies the following conditions:

$$\varphi(t) = n(t), \qquad \psi(x) = l(x). \tag{4.18}$$

As a result, we demonstrated the following statement.

Theorem 2. *An equation belonging to (1.1) can be transformed into the nonlinear Schrödinger equation (4.1) by an equivalence transformation of $G_{\mathcal{E}_{\chi_0}}$, if and only if the functions f, g and h are given by (4.14).*

Example. Let us consider the equation

$$\mathrm{i}u_t + k_1 t^{\frac{1}{2}} x^2 u_{xx} + k_2 t^{\frac{1}{2}} x^{-1}|u|^2 u + \tfrac{1}{4}k_1 t^{\frac{1}{2}} u = 0. \tag{4.19}$$

Taking into account that $\varphi(t) = t^{\frac{1}{2}}$ and $\psi(x) = x^{-1}$ and using the change of variables (4.15), we get

$$\hat{t} = \frac{2}{3}t\sqrt{t}, \qquad \hat{x} = \ln x, \qquad \hat{u}(\hat{t},\hat{x}) = \frac{u(t,x)}{\sqrt{x}}. \tag{4.20}$$

Rewriting (t, x, u) in terms of $(\hat{t}, \hat{x}, \hat{u})$, we obtain

$$t = \left(\frac{3}{2}\hat{t}\right)^{\frac{2}{3}}, \qquad x = \mathrm{e}^{\hat{x}}, \qquad u(t,x) = \mathrm{e}^{\frac{\hat{x}}{2}}\hat{u}(\hat{t},\hat{x}). \tag{4.21}$$

Using (4.21) one can transform (4.19) into (4.1). Equation (4.1), with $k_1 = k_2 = 1$, admits the following solution:

$$\hat{u}(\hat{t},\hat{x}) = \sqrt{2}a \exp\left[\mathrm{i}\frac{V_e}{2}\hat{x} + \mathrm{i}\left(a^2 - \frac{V_e^2}{4}\right)\hat{t}\right] \operatorname{sech}[a(\hat{x} - V_e\hat{t} - x_0)]. \tag{4.22}$$

Rewriting (4.22) in terms of the old variables (vide equation (4.20)) one gets

$$u(t,x) = \sqrt{2}a\sqrt{x}\exp\left[\mathrm{i}\frac{V_e}{2}\ln x + \mathrm{i}\left(a^2 - \frac{V_e^2}{4}\right)\frac{2t\sqrt{t}}{3}\right] \operatorname{sech}\left[a\left(\ln x - \frac{2}{3}V_e t\sqrt{t} - x_0\right)\right], \tag{4.23}$$

which is a solution of (4.19) with $k_1 = k_2 = 1$.

It is worth noting that a larger family of NLS equation (1.1) which can be mapped to the standard NLS equation could be obtained if one considers the equivalence algebra $\mathcal{E}$ when $\chi(t)$ is not constant.

Finally, we wish to stress that the presence of the arbitrary functions $n(t)$ and $l(x)$ in (4.14) provides enough freedom to recover variable coefficient NLS equations which may correspond to a particular physical situation.

5. Conclusions

In this paper we have obtained the infinitesimal generators of the equivalence transformations for the family of variable coefficient nonlinear Schrödinger equations (1.1). We have shown that the equivalence algebra $\mathcal{E}$ is an infinite dimensional one. Starting from a special equivalence subalgebra $\mathcal{E}_{\chi_0}$ and using the invariant test, we have shown that the family (1.1) does not admit zero-order differential invariants but admits the first-order ones. As an application of the differential invariants, we have characterized the most general NLS family of equation (1.1) that can be mapped to the standard NLS equation by an equivalence transformation of the subalgebra $\mathcal{E}_{\chi_0}$. The explicit form of this latter one has also been given explicitly.

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